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THE PROBLEM OF THE STABLE SYNTHESIS OF BOUNDED CONTROLS FOR A CERTAIN CLASS OF NON-STEADY SYSTEMS*

G.A. BESSONOV, V.I. KOROBOV and G.M. SKLYAR

Developing the results of /1-3/ with regard to the synthesis of bounded controls, a constructive method is given for constructing the controllability function and using the latter to set up a synthesizing control for a certain class of non-steady systems.

1. In this paper the method of Lyapunov functions is employed to solve the following problem of synthesizing bounded controls: given a controlled system

$$x^{\cdot} = f(t, x, u), \quad x \in R^n, \quad u \in \Omega \subset R^r \tag{1.1}$$

it is required to construct a control $u = u(t, x)$ satisfying a given constraint $u \in \Omega$ such that the trajectory $x(t)$ of system (1.1), beginning at an arbitrary point x_0 at time t_0, arrives at the final instant of time $t_0 + T\ (T = T(t_0, x_0))$ at a preassigned point x_1. The synthesis is said to be stable if x_1 is a rest point (i.e., $f(t, x_1, u_1) = 0$ for some $u_1 \in \Omega$ and any $t \in [t_0, t_0 + T)$) and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\| x(t) - x_1 \| < \varepsilon$ if $\| x_0 - x_1 \| < \delta$ and $t \in [t_0, t_0 + T)$. Otherewise, the synthesis is said to be unstable. Note that when x_1 is not a rest point the synthesis is, as a rule, unstable.

For example, consider the system $x_1^{\cdot} = x_2 + 1,\ x_2^{\cdot} = u,\ |u| \leqslant 1$. The requirement is that the trajectory reach the origin $O\ (x_1 = x_2 = 0)$ from an arbitrary point (x_1, x_2). The control solving the synthesis problem is: $u(x) = -1$ if $\varphi \geqslant 0$, $u(x) = 1$ if $\varphi < 0$, where $\varphi = x_1 + (x_2 + 1)\,\mathrm{sign}\,(x_2 + 1)/2$. However, any admissible synthesis in this problem is unstable. Indeed, let $x_1(t_0) > 0$. Then a necessary condition for reaching the origin is that at some time t_1, $x_1^{\cdot}(t_1) \leqslant 0$, i.e., $x_2(t_1) \leqslant -1$, whence it follows that any possible synthesis is unstable.

In this paper attention will be confined to the case of stable synthesis. Throughout the sequel it will be assumed, without loss of generality, that $x_1 = 0$. The control synthesis problem will be solved with the help of the controllability function $\Theta(t, x)$ /2/, which plays a role in the stable synthesis problem analogous to that of the Lyapunov function in stability theory.

2. Our solution of the synthesis problem is based on the following theorem.

Theorem 1. Consider the controlled process (1.1). Assume that the vector-function $f(t, x, u)$ is jointly continuous in all variables and, in the domain

$$\{(t, x, u)\colon t_0 \leqslant t \leqslant t_1,\ 0 < \rho_1 \leqslant \| x \| \leqslant \rho_2,\ u \in \Omega\}$$

satisfies a Lipschitz condition

$$\| f(t, x'', u'') - f(t, x', u') \| \leqslant L_1(\rho_1, \rho_2)(\| x'' - x' \| + \| u'' - u' \|)$$

Assume that in the closed domain

$$G = J \times \{x\colon \| x \| \leqslant R\} \quad (J = [t_0, t_1]) \tag{2.1}$$

where $0 < R \leqslant +\infty$, there exits a function $\Theta(t, x)$ satisfying the following conditions:

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1) $\Theta(t, x) > 0$ for $x \neq 0, t \in J$ and $\Theta(t, 0) = 0$ for any $t \in J$.

2) $\Theta(t, x)$ is continuous everywhere and continuously differentiable everywhere except possibly at points $(t, 0)$, $t \in J$.

3) There exists $c > 0$ such that the set $Q_c(t) = \{x: \Theta(t, x) \leqslant c\}$ is bounded and $Q_c(t) \subset \{x: \|x\| < R\}$ for all $t \in J$.

4) There exists a function $u(t, x) \in \Omega$ for $x \in Q_c(t)$, $t \in J$ such that

$$\frac{\partial \Theta(t, x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial \Theta(t, x)}{\partial x_i} f_i(t, x, u(t, x)) \leqslant -\beta \Theta^{1-1/\alpha}(t, x) \tag{2.2}$$

for some $\alpha > 0$ and $\beta > 0$; moreover, in the domain $\{(t, x): t_0 \leqslant t \leqslant t_1, 0 < \rho_1 \leqslant \|x\| \leqslant \rho_2\}$ the function satisfies a Lipschitz condition $\|u(t, x'') - u(t, x')\| \leqslant L_2(\rho_1, \rho_2)\|x'' - x'\|$.

5) $c \leqslant [\beta(t_1 - t_0)/\alpha]^\alpha$.

Then the trajectory of system (1.1) beginning at an arbitrary point $x_0 \in Q_c(t_0)$ at time t_0 will reach the point $x = 0$ at a certain final instant $t_0 + T$, where $T \leqslant \alpha\Theta^{1/\alpha}(t_0, x_0)/\beta$.

The proof of Theorem 1 is analogous to that of the corresponding theorem in /2/ for steady systems (1.1). Note that condition (2.2) guarantees the possibility of stable synthesis.

Remark 1. If $\alpha = +\infty$, then $\Theta(t, x)$ is a Lyapunov function guaranteeing asymptotic stability of the trivial solution of system (1.1).

In /1/, and also in /2, 3/, we devised methods for constructing a controllability function $\Theta(x)$ and a synthesizing control $u(x)$ for linear steady and certain classes of nonlinear ones. Unlike the Lyapunov function, however, the controllability function was exhibited in implicit form, i.e., as the solution of a certain algebraic or transcendental equation.

3. To illustrate the synthesis of bounded controls, we consider the system

$$\dot{x} = A(t)x + B(t)u, \quad x \in R^n, \quad u \in R^r, \quad \|u\| \leqslant d \tag{3.1}$$

We shall assume henceforth that

$$\begin{gathered} A(t),\ B(t) \in C^\infty[t_0, +\infty),\ \|A(t)\| \leqslant a, \quad \|\Delta^k B(t)\| \leqslant b^{k+1} \\ (k = 0, 1, \ldots), \quad \forall t \in [t_0, +\infty) \\ \Delta = A(t) - Id/dt \end{gathered} \tag{3.2}$$

(I is the identity matrix), and in addition

$$\operatorname{rank}(B(t), \Delta B(t), \ldots, \Delta^{n-1}B(t)) = n, \quad \forall t \in J \tag{3.3}$$

Theorem 2. Consider system (3.1). Let $\Theta(t, x)$ be the function defined by the equation

$$2a_0\Theta = (N_\Theta^{-1}(t)x, x), \quad x \neq 0, \quad a_0 > 0 \tag{3.4}$$

in the closed domain G (2.1); $\Theta(t, 0) = 0$, where

$$N_\Theta(t) = \int_t^\infty \exp\left(\frac{t-\tau}{\Theta}\right) \Phi(t, \tau) B(\tau) B^*(\tau) \Phi^*(t, \tau)\, d\tau \tag{3.5}$$

(the asterisk denotes transposition), $\Phi(t, \tau)$ is the Cauchy matrix of the system $\dot{x} = A(t)x$.

Then there exists $c > 0$ such that the set $Q_c(t) = \{x: \Theta(t, x) \leqslant c\}$ is bounded and $Q_c(t) \subset \{x: \|x\| < R\}$ for any $t \in J$; in addition, for any $x_0 \in Q_c(t_0) \setminus \{0\}$ the unique solution $x(t)$ of system (3.1) with the control

$$u(t, x) = -B^*(t) N_{\Theta(t,x)}^{-1}(t) x, \quad x \in Q_c(t) \setminus \{0\} \tag{3.6}$$

and initial condition $x(t_0) = x_0$ is defined on a certain semi-open interval $[t_0, t_0 + T) \subset J$ and satisfies the condition $\lim x(t) = 0$ as $t \to t_0 + T$, where $T \leqslant \Theta(t_0, x_0)/\beta$, $\beta > 0$. Moreover, $c \leqslant \beta(t_1 - t_0)$ and for any $d > 0$ the coefficient $a_0 > 0$ may be so chosen that the control $u(t, x)$ satisfies the constraint $\|u(t, x)\| \leqslant d$ for $x \in Q_c(t) \setminus \{0\}$, $t \in J$.

Remark 2. If Θ is fixed, then for any $\Theta \in (0, 1/(2a))$ the function (3.6) is a stabilizing control for system (3.1), and $v = (N_\Theta^{-1}(t)x, x)/(2a_0)$ is a Lyapunov function.

The proof of Theorem 2 amounts essentially to verifying that $u(t, x)$ is bounded and satisfies conditions 1 through 5 of Theorem 1 for the selected functions $\Theta(t, x)$ and $u(t, x)$.

Note that under the assumptions of Theorem 2, if $0 < \Theta < 1/(2a)$, the matrices $N_\Theta(t)$ and

$$N_{1\Theta}(t) = \int_t^\infty (\tau - t) \exp\left(\frac{t-\tau}{\Theta}\right) \Phi(t, \tau) B(\tau) B^*(\tau) \Phi^*(t, \tau)\, d\tau \tag{3.7}$$

are bounded and positive definite:

$$\| N_\Theta(t) \| \leqslant \frac{b^2\Theta}{1-2a\Theta}, \quad \| N_{1\Theta}(t) \| \leqslant \frac{b^2\Theta^2}{(1-2a\Theta)^2} \tag{3.8}$$

$(N_\Theta(t)x, x) > 0$, $(N_{1\Theta}(t)x, x) > 0$ for $x \neq 0$ and any $t \in J$. In addition, one can show by direct substitution that $N_\Theta(t)$, $N_{1\Theta}(t)$ and $N_\Theta^{-1}(t)$ satisfy the equations

$$dN_\Theta/dt - AN_\Theta - N_\Theta A^* = -BB^* + N_\Theta/\Theta \tag{3.9}$$

$$dN_{1\Theta}/dt - AN_{1\Theta} - N_{1\Theta}A^* = -N_\Theta + N_{1\Theta}/\Theta \tag{3.10}$$

$$dN_\Theta^{-1}/dt + N_\Theta^{-1}A + A^*N_\Theta^{-1} = N_\Theta^{-1}BB^*N_\Theta^{-1} - N_\Theta^{-1}/\Theta \tag{3.11}$$

We shall show that Eq.(3.4) has a unique positive solution $\Theta(t, x)$ in some closed domain G (2.1).

Consider the function $F(t, x, \Theta) = 2a_0\Theta - (N_\Theta^{-1}(t)x, x)$. Since

$$(N_\Theta^{-1}(t)x, x) \geqslant \| N_\Theta(t) \|^{-1} \| x \|^2 \geqslant (1 - 2a\Theta) \| x \|^2/(b^2\Theta)$$

(we are using the first inequality of (3.8)), it follows that $F(t, x, \Theta) < 0$ for sufficiently small Θ $(x \neq 0, t \in J)$. In addition, for all $x \neq 0, t \in J$ the function $F(t, x, \Theta)$ is monotone increasing since $\partial F(t, x, \Theta)/\partial\Theta \geqslant 2a_0 > 0$.

On the other hand, $F(t, x, \Theta) \geqslant 0$ for $\Theta = \lambda < 1/(2a)$ (where $\lambda = 1/(2a + \varepsilon), \varepsilon > 0$) and any $(t, x) \in G$, where

$$R = [2a_0\lambda/(\max_{t \in J} \| N_\lambda^{-1}(t) \|)]^{1/2}$$

This follows from the inequality

$$(N_\lambda^{-1}(t)x, x) \leqslant \| N_\lambda^{-1}(t) \| \| x \|^2 \leqslant \| N_\lambda^{-1}(t) \| R^2 \leqslant 2a_0\lambda$$

i.e., $F(t, x, \lambda) \geqslant 0$.

Thus, Eq.(3.4) defines a positive function $\Theta(t, x)$ in the domain G for $x \neq 0$. The fact that this function is continuous and continuously differentiable for $x \neq 0$ and any $t \in J$ follows from the implicit function theorem, since $\partial F(t, x, \Theta)/\partial\Theta \neq 0$.

Put $\Theta(t, 0) = 0, t \in J$. It can be shown that $\Theta(t, x)$ is continuous at $x = 0$ for any $t \in J$.

Indeed, suppose the contrary: there exists $\varepsilon_0 > 0$ such that, for any $\delta > 0$, there exist $x', \| x' \| < \delta$, and $t' \in J$ such that $\Theta(t', x') \geqslant \varepsilon_0$. We have

$$2a_0\Theta(t', x') = (N_{\Theta(t', x')}^{-1}(t')x', x') \leqslant \| N_{\Theta(t', x')}^{-1}(t') \| \| x' \|^2 \leqslant M\delta^2$$

$$(M = \max_{t \in J} \| N_{\varepsilon_0}^{-1}(t) \|)$$

If then $\delta < (a_0\varepsilon_0/M)^{1/2}$, we obtain $\Theta(t', x') < \varepsilon_0/2$, contrary to our assumption.

The inequality $(N_\Theta^{-1}(t)x, x) \geqslant \| N_\Theta^{-1}(t) \|^{-1} \| x \|^2$ implies that the set $Q_c(t)$ is bounded and $Q_c(t) \subset \{x: \| x \| < R\}$ for $c < R(\sqrt{a^2R^2 + 2a_0b^2} - aR)/(2a_0b^2)$ and any $t \in J$. Thus conditions 1-3 of Theorem 1 are satisfied.

To verify the other conditions of the theorem we need some auxiliary results.

Consider the operator D_Θ $(0 < \Theta < 1/a)$, defined for matrices $P(t)$ as follows:

$$(D_\Theta P)(t) = \int_t^\infty \exp\left(\frac{t-\tau}{\Theta}\right)\Phi(t, \tau)P(\tau)\,d\tau \tag{3.12}$$

It can be verified that D_Θ has the following properties:

1°. If $\| P(t) \| \leqslant b$ for $t \in [t_0, +\infty)$, then

$$\| (D_\Theta^m P)(t) \| \leqslant b\,[\Theta/(1 - a\Theta)]^m$$

2°. If $P(t) \in C^1[t_0, +\infty)$, $\| \Delta P(t) \| \leqslant b$ for $t \in [t_0, +\infty)$, then $\Theta^{-m}(D_\Theta^m P)(t) \rightrightarrows P(t)$ as $\Theta \to 0$ on $[t_0, +\infty)$.

Lemma 1. There exists $\Theta_0 > 0$ such that for $0 < \Theta < \Theta_0$

$$N_\Theta(t) = \frac{1}{\Theta}\sum_{k=0}^{\infty} E_k^{(1)}(t, \Theta) \tag{3.13}$$

$$N_{1\Theta}(t) = \frac{1}{\Theta^2}\sum_{k=0}^{\infty} (1 + k)E_k^{(2)}(t, \Theta) \tag{3.14}$$

$$(E_k^{(n)}(t, \Theta) = \{(D_\Theta^n(\Delta D_\Theta)^k B)(t)\}\{(D_\Theta^n(\Delta D_\Theta)^k B)(t)\}^*, \quad n = 1, 2)$$

Proof. It follows from the properties of the operator D_Θ that the series (3.13) and (3.14) may be differentiated with respect to t on $[t_0, +\infty)$ for $0<\Theta<1/(a+b)$. Hence, one can show by direct substitution that these series satisfy Eqs.(3.9) and (3.10), respectively.

Thus (3.5) and (3.13) are bounded solutions of Eq.(3.9), and (3.7) and (3.14) are bounded solutions of Eq.(3.10), $t \in [t_0, +\infty)$.

We transform Eq.(3.9) to the form

$$\frac{dN_\Theta}{dt} + \left[-\left(A + \frac{1}{2\Theta} I\right)\right] N_\Theta + N_\Theta\left[-\left(A + \frac{1}{2\Theta}\right) I\right]^* + BB^* = 0 \tag{3.15}$$

Let us assume that Eq.(3.9), and hence also (3.15), has two distinct bounded solutions $N_\Theta^{(1)}(t)$ and $N_\Theta^{(2)}(t)$. Then their difference $K_\Theta(t) = N_\Theta^{(1)}(t) - N_\Theta^{(2)}(t)$ will satisfy the equation obtained from (3.15) by dropping the term BB^*. Consequently, the solutions $x(t)$ of the system $x^\cdot = -(A(t) + I/(2\Theta))x$ satisfy the identity

$$(K_\Theta(t)x, x) = c, \ \forall t \in [t_0, +\infty) \tag{3.16}$$

Choose q so that for $0<\Theta<q$ one has $\|x(t)\| \to 0$ as $t \to +\infty$. Then the identity (3.16) can hold for all $t \in [t_0, +\infty)$ only if $K_\Theta(t) \equiv 0$ for any $t \in [t_0, +\infty)$ and $c = 0$. Thus, when $0<\Theta<q$ Eq.(3.15), and hence also (3.9), has a unique bounded solution. Hence it follows that when $0<\Theta<\Theta_0 = \min(1/(2a);\ 1/(a+b);\ q)$ the matrix $N_\Theta(t)$ can be represented by the series (3.13) for any $t \in [t_0, +\infty)$.

Analogous arguments show that if $0<\Theta<q$ Eq.(3.10) has a unique bounded solution, and so, when $0<\Theta<\Theta_0$ the matrix $N_{1\Theta}(t)$ can be represented by the series (3.14) for any $t \in [t_0, +\infty)$.

Remark 3. The essential point in Eqs.(3.13) and (3.14) is the decomposition of the matrix into a sum of positive matrices. In the steady case these decompositions are simplified:

$$E_k^{(n)}(t, \Theta) = R_{1/\Theta}^n (R_{1/\Theta}A)^k BB^* (R_{1/\Theta}A)^{*k} R_{1/\Theta}^{*n}$$
$$R_{1/\Theta} = (A + I/\Theta)^{-1}$$

these formulae are used in the proof of Theorem 2 of /3/.

Lemma 2. There exists $\Theta_1 > 0$ such that for $0<\Theta<\Theta_1$

$$\Theta(N_\Theta(t)x, x) \geq \gamma_1 \Theta^{2n} \|x\|^2, \quad \gamma_1 > 0, \quad \forall t \in J \tag{3.17}$$

Proof. We have

$$\left(\sum_{k=0}^{n-1} P_k(t)x, x\right) = (G_n(t)G_n^*(t)x, x) = (G_n^*(t)x, G_n^*(t)x) > 0$$
$$(P_k(t) = (\Delta^k B(t))(\Delta^k B(t))^*, \ G_n(t) = (B(t), \Delta B(t), \ldots, \Delta^{n-1}B(t)))$$

for $x \neq 0$ and any $t \in J$, since otherwise (3.3) would fail to hold. In addition, the function $G_n(t)$ is continuous, and so there exists a function $\gamma(t) > 0$ continuous on J such that $(G_n^*(t)x, G_n^*(t)x) \geq 2\gamma(t)\|x\|^2$. Therefore

$$\left(\sum_{k=0}^{n-1} P_k(t)x, x\right) \geq 2\gamma_1 \|x\|^2, \quad \gamma_1 = \min_{t \in J} \gamma(t) > 0 \tag{3.18}$$

It follows from the properties of D_Θ that

$$\sum_{k=0}^{n-1} \Theta^{-2(k+1)} E_k^{(1)}(t, \Theta) \rightrightarrows \sum_{k=0}^{n-1} P_k(t)$$

as $\Theta \to 0$ on J. Hence there exists $\Theta_1 > 0$ such that, for $0<\Theta<\Theta_1$

$$\left(\sum_{k=0}^{n-1} \Theta^{-2(k+1)} E_k^{(1)}(t, \Theta)x, x\right) \geq \gamma_1 \|x\|^2, \quad \forall t \in J$$

We shall assume that $\Theta_1 < 1$; inequality (3.17) then follows from the preceding inequality.

We can now prove the validity of condition 4 of Theorem 1.

Differentiating the relationship $2a_0\Theta(t, x) - (N_{\Theta(t,x)}^{-1}(t)x, x) = 0$ along trajectories of system (3.1) with a control of type (3.6) and noting (3.4) and (3.11), we obtain

$$\Theta^{\cdot}(t,x) \leqslant -\left[1+\frac{(N_{1\Theta}(t)y,y)}{\Theta(t,x)(N_{\Theta}(t)y,y)}\right]^{-1}, \quad y=N_{\Theta(t,x)}^{-1}(t)x \tag{3.19}$$

From Lemma 1 and the properties of D_Θ we obtain the inequality

$$(N_{1\Theta}(t)y,y) \leqslant 4[n\Theta(N_\Theta(t)y,y)+S], \qquad S=\Big(\sum_{k=n}^{\infty}(1+k)E_k^{(1)}(t,\Theta)y,y\Big) \tag{3.20}$$

Next, since $S \leqslant r_1\Theta^{2n}\|y\|^2$, $r_1>0$, for $0<\Theta\leqslant\Theta_2<1/(a+b)$, it follows from (3.20), Lemma 2 and (3.19) that

$$\Theta^{\cdot}(t,x) \leqslant -[1+4(n+r_1/\gamma_1)]^{-1}, \quad x\neq 0$$

in the closed domain G (2.1), where

$$R=(2a_0\Theta_4/\max_{t\in J}\|N_{\Theta_4}^{-1}(t)\|)^{1/2}, \quad 0<\Theta\leqslant\Theta_4<\Theta_3=\min(\Theta_0,\Theta_1,\Theta_2,1)$$

We will now show that the control is bounded for $x\in Q_c(t)\setminus\{0\}$, $t\in J$.

Using (3.6) and (3.1) and recalling the properties of D_Θ and Lemma 1, we have

$$\|u(t,x)\|^2 = 2a_0V_0(t,y(t,x),\Theta(t,x)) \tag{3.21}$$

Here

$$V_s(t,y,\Theta)=(M_s(t,\Theta)y,y)\Big(\sum_{k=0}^{\infty}E_k^{(1)}(t,\Theta)y,y\Big)^{-1}, \quad s=0,1,\ldots \tag{3.22}$$
$$M_s(t,\Theta)=\{((\Theta\Delta+I)D_\Theta(\Delta D_\Theta)^sB)(t)\}\{((\Theta\Delta+I)D_\Theta(\Delta D_\Theta)^sB)(t)\}^*$$

Consider the family of functions (3.22) for $0<\Theta<\Theta_3$, $y\neq 0$, $t\in J$. Using the properties of D_Θ, one can show that $V_s(t,y,\Theta)\leqslant 2+V_{s+1}(t,y,\Theta)$. Therefore

$$V_0(t,y,\Theta)\leqslant 2^{s+1}-2+2^sV_s(t,y,\Theta), \quad s=1,2,\ldots$$

Now, for $0<\Theta\leqslant\Theta_4<\Theta_3$,

$$(M_{n-1}(t,\Theta)y,y)\leqslant\left(\frac{b^{n+1}\Theta_4+b^n}{1-a\Theta_4^n}\right)^2\Theta^{2n}\|y\|^2=m_1\Theta^{2n}\|y\|^2$$

and so, in view of Lemma 2, we infer that $V_{n-1}(t,y,\Theta)\leqslant m_1/\gamma_1$. Choosing

$$a_0<d^2/(2^{n+1}-4+2^n m_1/\gamma_1)$$

we see that the control is bounded:

$$\|u(t,x)\|\leqslant d, \quad x\in Q_{\Theta_4}(t)\setminus\{0\}, \quad t\in J$$

Now, if we require that

$$c<\min\{R\sqrt{a^2R^2+2a_0b^2}-aR;\ \beta(t_1-t_0);\ \Theta_4\}$$

where $\beta=(1+4(n+r_1/\gamma_1))^{-1}$, then

$$Q_c(t)\subset\{x\colon \|x\|<R\} \text{ and } t_0+T\leqslant t_1.$$

This completes the proof of Theorem 2.

Note that the assumptions of Theorem 2 are satisfied, for example, if the elements of the matrix $\Phi^{-1}(t)B(t)$ are quasipolynomials.

Example. Consider the system

$$x_1^{\cdot}=\frac{x_1}{t}+\frac{2x_2}{t^2}+\frac{u}{t}, \quad x_2^{\cdot}=-3x_1-\frac{3x_2}{t}+u; \quad |u|\leqslant 1, \quad t\in[1;10]$$

Then

$$(B(t),\Delta B(t))=\begin{Vmatrix}1/t & 2/t^2\\ 1 & -6/t\end{Vmatrix}, \quad \Phi(t)=\begin{Vmatrix}1/t & -2/t^2\\ -1 & 3/t\end{Vmatrix}$$

Suppose, say, that $b=7.22$, $a=4.8$, $0<\Theta<0.707$, $\gamma_1=3.66\cdot10^{-3}$, $m_1=52$, $r_1=0$, $\beta=1/9$, $a_0=1.7\cdot10^{-5}$. Then the control solving the synthesis problem has the form

$$u(t,x)=-\left(\frac{t^2}{\Theta^2}+\frac{12t}{\Theta}\right)\frac{x_1}{10}-\left(\frac{t}{\Theta^2}-\frac{8}{\Theta}\right)\frac{x_2}{10}$$

$$x \in Q_c(t) \setminus \{0\} = \{x: \Theta \leqslant 4.47 \cdot 10^{-3}\} \setminus \{0\}, \ t \in [1; 10]$$

where $\Theta = \Theta(t, x)$ is the unique positive solution of the equation

$$3.4 \cdot 10^{-5}\Theta^4 - (72t^2x_1^2 + 96tx_1x_2 - 32x_2^2)\Theta^2 + (12t^3x_1^2 - 4t^2x_1x_2 - 8tx_2^2)\Theta + t^4x_1^2 - 2t^3x_1x_2 = t^2x_2^2$$

4. Consider system (3.1) (to simplify the discussion we assume that $r = 1$). Define an n-dimensional vector-function $c(t)$ from the conditions $c^*(t)\,\Delta^i B(t) = 0, i = 0, 1, \ldots, n-2$; $c^*(t)\,\Delta^{n-1}B(t) = 1$. In view of (3.3) and the choice of $c(t)$, it can be shown that the matrix

$$H(t) = \operatorname{col}(c^*(t),\ (\Delta_1 c(t))^*, \ldots, (\Delta_1^{n-1} c(t))^*) \qquad (\Delta_1 = A^*(t) + I d/dt) \tag{4.1}$$

is non-singular for any $t \in J$. Transforming in (3.1) to the new variable $z = H(t)x$ and introducing a new control $v = (\Delta_1^n c(t))^* H^{-1}(t) z + u$, we obtain a steady system

$$z_i^{\cdot} = z_{i+1}, \quad i = 1, \ldots, n-1; \quad z_n^{\cdot} = v, \quad |v| \leqslant d_1 < d \tag{4.2}$$

Solving the control synthesis problem for this system, using Theorem 2 or Theorem 2 of /3/ and returning to the old variables and control, we obtain

Theorem 3. Let $\Theta(t, x)$ be the function defined by the equation

$$2a_0\Theta^{2n} = \sum_{i,j=1}^{n} f_{ij}\Theta^{i+j-2}\,[(\Delta_1^{i-1}C(t))^* x]\,[(\Delta_1^{j-1}C(t))^* x], \quad x \neq 0 \tag{4.3}$$
$$(N_\Theta^{-1} = \| f_{ij}\Theta^{i+j-2n-1} \|_{i, j=1, \ldots, n})$$

in the closed domain G (2.1), $\Theta(t, 0) = 0$.

Then there exists $c > 0$ such that $Q_c(t)$ is bounded and $Q_c(t) \subset \{x: \ \|x\| < R\}$ for any $t \in J$, and for any $x_0 \in Q_c(t_0) \setminus \{0\}$ the unique solution $x(t)$ of system (3.1) with control

$$u(t, x) = -\sum_{j=1}^{n} \frac{f_{nj}(\Delta_1^{j-1}C(t))^* x}{\Theta^{n-j+1}} - (\Delta_1^{n}C(t))^* x, \quad x \in Q_c(t) \setminus \{0\} \tag{4.4}$$

and initial condition $x(t_0) = x_0$ is defined on some semiclosed interval $[t_0, t_0 + T) \subset J$ and satisfies the condition: $\lim x(t) = 0$ as $t \to t_0 + T$, where $T \leqslant (1 + 4n)\,\Theta(t_0, x_0)$. Moreover, $c \leqslant (t_1 - t_0)/(1 + 4n)$ for $\|u(t, x)\| \leqslant d$ for $x \in Q_c(t) \setminus \{0\}, t \in J$.

Example. Consider the motion of a rigid body with a single axis of symmetry controlled by two jet engines, described by the equations

$$x_1^{\cdot} = -ax_2x_3 + u_1\cos\omega t, \quad x_2^{\cdot} = ax_1x_3 - u_1\sin\omega t, \quad x_3^{\cdot} = u_2 \ |u_1| \leqslant 1, \quad |u_2| \leqslant 1, \quad a = \text{const}$$

Here x_1, x_2, x_3 are the projections of the angular velocity of the body on coordinate axes rigidly attached to it, ω is the angular velocity of rotation of the first engine, and u_1 and u_2 are the controlling torques.

The problem is to construct controls $u_1(t, x)$ and $u_2(t, x)$ that will take the body from an arbitrary position $x(0) = x_0$ to the point O $(x_1 = x_2 = x_3 = 0)$ in a finite time.

Choose $u_2(t, x) = -\operatorname{sign} x_{30}$. Then the system becomes linear. Using Theorem 3, we see that the control

$$u_1(t, x) = \frac{\xi}{\Theta^2\varphi} - \frac{2}{\Theta}\left(\eta + \frac{\xi}{\varphi^2}\right) + \varphi\xi + \frac{\eta}{\varphi}$$
$$u_2(t, x) = -\operatorname{sign} x_{30}; \quad \varphi = t\operatorname{sign} x_{30} - x_{30} - \omega$$
$$\xi = x_1\sin\omega t + x_2\cos\omega t, \quad \eta = x_1\cos\omega t - x_2\sin\omega t$$

where $\Theta = \Theta(t, x)$ is defined by the equation

$$a_0\Theta^4 = \left[\left(\eta + \frac{\xi}{\varphi^2}\right)\Theta - \frac{\xi}{2\varphi}\right]^2 + \frac{\xi}{4\varphi^2}$$

takes any point x_0 from some given neighbourhood of the origin to the point O in a finite time $T = |x_{30}|$, and moreover $|u_i(t, x)| \leqslant 1, \ i = 1, 2$.

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EXPONENTIAL STABILIZATION OF NON-LINEAR STOCHASTIC SYSTEMS*

V.A. UGRINOVSKII

We consider the stabilization of non-linear systems whose parameters are subjected to "white noise". For stochastic systems with non-linear feedback, we derive sufficient (frequency-domain) conditions of exponential stabilization by a controller that uses information about the system output (incomplete system state information). The problem of the stabilization of linear stochastic systems has been studied in some detail /1-3/. Yet for non-linear stochastic systems we only have general theorems that reduce the stabilization problem to finding a stochastic Lyapunov function /4, 5/.

In this paper, we derive sufficient conditions of exponential stabilization by methods of the theory of absolute stochastic stability. The advantages of these methods are well-known: the specific Lyapunov function is not required, and its existence in the class of functions "quadratic form plus integrals over non-linearities" is easily checked /6/. The latest results of this theory for stochastic systems /7/ make it possible to solve the stabilization problem for a wide class of non-linear systems with parametric disturbances.

1. Formulation of the problem.

We consider a controllable dynamic system described by Ito's differential equation

$$x^{\cdot} = (A_0 + \Sigma A_j w_j^{\cdot}) x + (b_0 + \Sigma b_j w_j^{\cdot}) u + (q_0 + \Sigma q_j w_j^{\cdot}) \varphi(\sigma), \quad \sigma = \nu^* x \tag{1.1}$$

Here x is the n-dimensional state vector, u is the d-dimensional control vector, σ is the l-dimensional vector of observed variables, φ is the m-dimensional vector function describing the non-linear feedback or allowing for other non-linear effects in the system, A_j, b_j, q_j $(j = 0, 1, \ldots, s)$ are appropriately dimensioned constant matrices, and w_j $(j = 1, \ldots, s)$ are independent standard Wiener processes; here and henceforth, summation is over j from $j = 1$ to $j = s$, unless otherwise stated.

The class of admissible non-linear functions $\varphi(\sigma)$ is described in accordance with the general theory of absolute stability /6/. Let

$$F_1(\sigma, x, \varphi, \psi) = \sigma^* r\sigma + 2\sigma^* p\varphi + \varphi^* g\varphi - \Sigma f_j^* \theta\psi \tag{1.2}$$
$$f_j = \operatorname{diag}[\Lambda_j\Lambda_j^*], \quad \Lambda_j = \nu^*(A_j x + q_j\varphi), \quad j = 1, \ldots, s$$

The symbol $\operatorname{diag}[\cdot]$ is the vector formed from the main diagonal elements of the matrix in brackets: f_j $(j = 1, \ldots, s)$ are l-dimensional vectors, and ψ is an m-dimensional vector. The real matrices $r = r^*$, p, $g = g^*$, θ are $l \times l$, $l \times m$, $m \times m$, $l \times m$ respectively. We assume that the matrix θ satisfies the following conditions: a) it is non-zero only when $\nu^* b_j = 0$ for all $j = 1, \ldots, s$; b) if condition a) holds, then the element θ_{kl} of the matrix θ may be non-zero only if φ_l is a continuously differentiable function of a single variable σ_k the k-th component of the vector σ.

We assume that the non-linearity φ satisfies the condition

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